

# Large-gradient and Lagrange singularities of solutions of a quasilinear parabolic equation

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**Abstract.** Asymptotic solutions of a quasilinear parabolic equation with a small parameter at the higher derivative are constructed near large-gradient and Lagrange singularities of  $A$ -type, which represent interest for studying processes of shock waves formation in physical media with a small nonzero viscosity.

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## 1 Introduction

A simplest model of the motion of continuum, which takes into account nonlinear effects and dissipation, is the equation of nonlinear diffusion

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} = \varepsilon \frac{\partial^2 u}{\partial x^2}$$

for the first time presented by H. Bateman [1] and J. Burgers [2]. This equation is used in studying the evolution of a wide class of physical systems and probabilistic process, acoustic waves in fluid and gas [3].

In the present paper, results of investigations of the asymptotic behavior of solutions near singular points in the Cauchy problem for the more general quasilinear parabolic equation

$$\frac{\partial u}{\partial t} + \frac{\partial \varphi(u)}{\partial x} = \varepsilon \frac{\partial^2 u}{\partial x^2}, \quad t \geq t_0, \quad (1)$$

$$u(x, t_0) = q(x), \quad x \in \mathbb{R}, \quad (2)$$

are given. We assume that  $\varepsilon > 0$ , the function  $\varphi$  is infinitely differentiable and its second derivative is strictly positive. The initial function  $q$  is bounded and smooth.

The interest to study the behavior of solutions near singular points is explained, in particular, the fact that such singular events last very little time and, however, they determine all subsequent behavior of the system in many respects. The asymptotic behavior of solutions in neighborhoods of singular points is directly connected with constructing approximations in neighborhoods of shock waves, which is important for applied problems in the mechanics of continua.

Although the types of singular points of solutions are in detail classified [4] and processes of shock waves formation for the degenerate first-order equation are studied [5], constructing asymptotic series in the viscosity parameter  $\varepsilon$  for an equation of a more general form (1) in every specific case is a separate and very complex problem. For example, in the problem about the transition of a weak discontinuity into a strong one [6] there is still an open question of matching asymptotic series.

## 2 Singularity generated by a large initial gradient

First, we consider a singular point of the solution with two small parameters [7] in the case, when the initial condition has the form

$$u(x, 0, \varepsilon, \rho) = \nu(x\rho^{-1}), \quad x \in \mathbb{R}, \quad \rho > 0,$$

where function  $\nu$  is infinitely differentiable and bounded, and  $\rho$  is the second small parameter. It is proved that in this case under conditions

$$\nu(\sigma) = \sum_{n=0}^{\infty} \frac{\nu_n^{\pm}}{\sigma^n}, \quad \sigma \rightarrow \pm\infty, \quad (\nu_0^- > \nu_0^+)$$

for the solution of problem (1)–(2) as

$$\varepsilon \rightarrow 0 \quad \text{and} \quad \mu = \rho/\varepsilon \rightarrow 0$$

in the strip

$$\{(x, t) : x \in \mathbb{R}, \quad 0 \leq t \leq T\}$$

there holds the asymptotic formula

$$u(x, t, \varepsilon, \rho) = h_0\left(\frac{x}{\rho}, \frac{\varepsilon t}{\rho^2}\right) - R_{0,0,0}\left(\frac{x}{2\sqrt{\varepsilon t}}\right) + \Gamma\left(\frac{x}{\varepsilon}, \frac{t}{\varepsilon}\right) + O(\mu^{1/2} \ln \mu),$$

where

$$h_0(\sigma, \omega) = \frac{1}{2\sqrt{\pi\omega}} \int_{-\infty}^{\infty} \nu(s) \exp\left[-\frac{(\sigma-s)^2}{4\omega}\right] ds,$$

$$R_{0,0,0}(z) = \nu_0^- \operatorname{erfc}(z) + \nu_0^+ \operatorname{erfc}(-z),$$

$$\operatorname{erfc}(z) = \frac{1}{\sqrt{\pi}} \int_z^{+\infty} \exp(-y^2) dy,$$

$$\sigma = \frac{x}{\rho}, \quad \omega = \frac{\varepsilon t}{\rho^2}, \quad z = \frac{\sigma}{2\sqrt{\omega}},$$

the function  $\Gamma$  is the solution of the equation in the inner variables

$$\eta = x/\varepsilon, \quad \theta = t/\varepsilon,$$

$$\frac{\partial \Gamma}{\partial \theta} + \frac{\partial \varphi(\Gamma)}{\partial \eta} - \frac{\partial^2 \Gamma}{\partial \eta^2} = 0$$

with the initial condition

$$\Gamma(\eta, 0) = \begin{cases} \nu_0^-, & \eta < 0, \\ \nu_0^+, & \eta > 0. \end{cases}$$

In addition in [8], using the renormalization method [9] the following asymptotic formula is obtained:

$$u(x, t, \varepsilon, \rho) = \frac{1}{\nu_0^+ - \nu_0^-} \int_{-\infty}^{\infty} \Gamma\left(\frac{x - \rho s}{\varepsilon}, \frac{t}{\varepsilon}\right) \nu'(s) ds + O(\mu^{1/4}).$$

### 3 Fold singularity

Consider the Cauchy problem (1)–(2) for the quasilinear parabolic equation when the solution of the limit problem ( $\varepsilon = 0$ ) has a point of gradient catastrophe. A.M. Il'in studied this problem in the case when in the strip

$$\{(x, t) : t_0 \leq t \leq T, x \in \mathbb{R}\}$$

the limit solution is a smooth function everywhere except for one smooth discontinuity curve

$$\{(x, t) : x = s(t), t \geq t^* > t_0\}.$$

A detailed presentation of his results can be found in monograph [10], where the asymptotics of the solution as  $\varepsilon \rightarrow 0$  is constructed and justified with an arbitrary accuracy. Under a suitable choice of independent variables, the singular point  $(s(t^*), t^*)$  coincides with the origin and in its neighborhood the following stretched variables are introduced:

$$\xi = \varepsilon^{-3/4}x, \quad \tau = \varepsilon^{-1/2}t.$$

An expansion of the solution is sought in the form of the series

$$w = \sum_{k=1}^{\infty} \varepsilon^{k/4} \sum_{j=0}^{k-1} w_{k,j}(\xi, \tau) \ln^j \varepsilon^{1/4}.$$

Substituting it into equation (1), for coefficients  $w_{k,j}$  we obtain the recurrence system

$$\begin{aligned} \frac{\partial w_{1,0}}{\partial \tau} + \varphi''(0)w_{1,0} \frac{\partial w_{1,0}}{\partial \xi} - \frac{\partial^2 w_{1,0}}{\partial \xi^2} &= 0, \\ \frac{\partial w_{k,j}}{\partial \tau} + \varphi''(0) \frac{(\partial w_{1,0} w_{k,j})}{\partial \xi} - \frac{\partial^2 w_{k,j}}{\partial \xi^2} &= E_{k,j}, \quad k \geq 2. \end{aligned}$$

These equations should be supplied with the conditions

$$w_{k,j}(\xi, \tau) = W_{k,j}(\xi, \tau), \quad \tau \rightarrow -\infty,$$

where  $W_{k,j}(\xi, \tau)$  is the sum of all coefficients at  $\varepsilon^{k/4} \ln^j \varepsilon^{1/4}$  in the reexpansion of the asymptotics far from the singularity (the outer expansion) in terms of the inner variables.

Investigation of solutions of this system is the central and most laborious task in this problem. It is proved that there exist solutions  $w_{k,j}(\xi, \tau)$  for  $k \geq 2$ ,  $0 \leq j \leq k-1$ , which are infinitely differentiable for all  $\xi$  and  $\tau$ .

Observe separately the properties of the leading term, which is found with the help of the Cole–Hopf transform

$$w_{1,0}(\xi, \tau) = -\frac{2}{\varphi''(0)\Lambda(\xi, \tau)} \frac{\partial \Lambda(\xi, \tau)}{\partial \xi},$$

where

$$\Lambda(\xi, \tau) = \int_{-\infty}^{\infty} \exp(-2s^4 + \tau s^2 - \xi s) ds$$

is a real-valued analog of the Pearcey function [11], which is a solution of the heat equation. The argument of the exponent is the generating family of the Lagrange singularity  $A_3$ , see [4].

**Theorem 1.** *The function  $w_{1,0}$  satisfies the asymptotic relations*

$$w_{1,0}(\xi, \tau) = [\varphi''(0)]^{-1} H(\xi, \tau) + \sum_{l=1}^{\infty} h_{1-4l}(\xi, \tau), \quad 3[H(\xi, \tau)]^2 - \tau \rightarrow \infty,$$

$$(\xi, \tau) \in \Omega_1 = \mathbb{R}^2 \setminus \{|\xi| < \tau^{\gamma_1-1/2}, \tau > 0, 0 < \gamma_1 < 2\},$$

where  $H(\xi, \tau)$  is the Whitney fold function,

$$H^3 - \tau H + \xi = 0,$$

$h_l(\xi, \tau)$  are homogeneous functions of power  $l$  in  $H(\xi, \tau)$ ,  $\sqrt{-\tau}$  and  $\sqrt{3[H(\xi, \tau)]^2 - \tau}$ , which are polynomials in  $H(\xi, \tau)$ ,  $\tau$  and  $(3[H(\xi, \tau)]^2 - \tau)^{-1}$ ,

$$w_{1,0}(\xi, \tau) = \sqrt{\tau} \left( -\frac{\text{th } z}{\varphi''(0)} + \sum_{k=1}^{\infty} \tau^{-2k} q_k(z) \right), \quad \tau \rightarrow +\infty,$$

$$(\xi, \tau) \in \Omega_2 = \{|\xi| \tau^{1/2} < \tau^{\gamma_2}, \gamma_1 < \gamma_2 < 2\},$$

where  $z = \xi \sqrt{\tau}/2$ ,  $|q_k(z)| \leq M_k(1 + |z|^k)$ ,  $q_k \in C^\infty(\mathbb{R}^1)$  are solutions of a recurrence system of ordinary differential equations.

The proof of the theorem is based on calculating the asymptotics of the integral  $\Lambda(\xi, \tau)$  by Laplace's method. In the domain  $\Omega_1$  the contribution into the asymptotics is given by one local maximum and in the domain  $\Omega_2$  by two local maxima.

## 4 Lagrange singularity $A_{2n+1}$

It is investigated the problem with such an initial function  $q(x)$  that

$$\varphi'(q(x)) = -x + x^3 + O(x^4), \quad x \rightarrow 0.$$

In paper [12], the more general condition

$$\varphi'(q(x)) = -x + x^{2n+1} + O(x^{2n+2}), \quad x \rightarrow 0,$$

with an arbitrary natural  $n$  has been considered. Then in the Cole–Hopf transform instead of the Pearcey function one should consider the integral

$$\int_{-\infty}^{\infty} \exp(-as^{2n+2} + \theta s^2 - \eta s) ds$$

corresponding a section of the Lagrange singularity  $A_{2n+1}$ .

Let us clear out scales of the inner variables, which are introduced using the change

$$x = \eta \varepsilon^\sigma, \quad t = \theta \varepsilon^\mu, \quad u \sim \varepsilon^\kappa.$$

Since all terms in equation (1) should be of the same order, we obtain the relation

$$-\mu = \kappa - \sigma = 1 - 2\sigma.$$

From the characteristic equation

$$x = y + (t + 1)\varphi'(q(y))$$

we obtain another relation

$$\sigma = \kappa + \mu = (2n + 1)\kappa.$$

From these relations we find

$$\sigma = \frac{2n+1}{2n+2}, \quad \mu = \frac{n}{n+1}, \quad \kappa = \frac{1}{2n+2}. \quad (3)$$

Since  $u = O(\varepsilon^{1/(2n+2)})$ , equation (1) becomes the Burgers equation, whose solution can be written in the form of the Cole–Hopf transform. Moreover, the coefficient at  $s^{2n+2}$  is determined from the condition of matching the inner asymptotics and the outer expansion

$$u_{\text{out}} \sim \frac{\varepsilon^\kappa U_0(\eta, \theta)}{\varphi''(0)}, \quad U_0^{2n+1} - \theta U_0 + \eta = 0.$$

**Theorem 2.** *In the domain  $\Omega_\varepsilon = \{(x, t) : |x\varepsilon^{-\kappa}| + |t| < K\varepsilon^\mu, K > 0\}$  for any  $n \geq 2$  the function*

$$u_{\text{in}}(x, t, \varepsilon) = -2\varepsilon[\varphi''(0)V(x, t, \varepsilon)]^{-1} \frac{\partial V(x, t, \varepsilon)}{\partial x},$$

where

$$V(x, t, \varepsilon) = \int_{-\infty}^{\infty} \exp\left(-\frac{2^{2n}s^{2n+2}}{n+1} + \frac{ts^2}{\varepsilon^\mu} - \frac{xs}{\varepsilon^\sigma}\right) ds,$$

$$\sigma = \frac{2n+1}{2n+2}, \quad \mu = \frac{n}{n+1},$$

is an asymptotic solution of equation (1) in the following sense:

$$\frac{\frac{\partial u_{\text{in}}}{\partial t} + \frac{\partial \varphi(u_{\text{in}})}{\partial x} - \varepsilon \frac{\partial^2 u_{\text{in}}}{\partial x^2}}{\sup_{(x,t) \in \Omega_\varepsilon} \left\{ \left| \frac{\partial u_{\text{in}}}{\partial t} \right| + \left| \frac{\partial \varphi(u_{\text{in}})}{\partial x} \right| + \left| \varepsilon \frac{\partial^2 u_{\text{in}}}{\partial x^2} \right| \right\}} = O(\varepsilon^\kappa), \quad (4)$$

where

$$\varkappa = \frac{1}{2n+2}.$$

**Proof.** From the equation

$$\frac{\partial V}{\partial t} = \varepsilon \frac{\partial^2 V}{\partial x^2}$$

it follows that

$$\frac{\partial u_{\text{in}}}{\partial t} + \varphi''(0)u_{\text{in}} \frac{\partial u_{\text{in}}}{\partial x} = \varepsilon \frac{\partial^2 u_{\text{in}}}{\partial x^2}.$$

Then, using the relation

$$\varphi'(u) = \varphi''(0)u + O(u^2),$$

we obtain

$$\frac{\partial u_{\text{in}}}{\partial t} + \frac{\partial \varphi(u_{\text{in}})}{\partial x} - \varepsilon \frac{\partial^2 u_{\text{in}}}{\partial x^2} = \varphi'(u_{\text{in}}) \frac{\partial u_{\text{in}}}{\partial x} - \varphi''(0)u_{\text{in}} \frac{\partial u_{\text{in}}}{\partial x} = O\left(u_{\text{in}}^2 \frac{\partial u_{\text{in}}}{\partial x}\right).$$

Since

$$u_{\text{in}}^2 = 4\varepsilon^{2-2\sigma}[\varphi''(0)V]^{-2}(V_\eta)^2,$$

$$\frac{\partial u_{\text{in}}}{\partial x} = -2\varepsilon^{1-2\sigma}[\varphi''(0)]^{-1} \left( \frac{V_{\eta\eta}}{V} - \frac{(V_\eta)^2}{V^2} \right),$$

we find

$$\frac{\partial u_{\text{in}}}{\partial t} + \frac{\partial \varphi(u_{\text{in}})}{\partial x} - \varepsilon \frac{\partial^2 u_{\text{in}}}{\partial x^2} = O(\varepsilon^{3-4\sigma}).$$

Taking into account (3) and the order  $O(\varepsilon^{2-3\sigma})$  of derivatives on the left-hand side, we obtain estimate (4).

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